# Varianta 5

## Subiectul I

1. $A = \{x \in \mathbb{Z} \mid |x+1| \leq 2\}$

$|x+1| = \begin{cases} x+1, & x \geq -1 \Rightarrow x \in [-1, \infty) \\ -x-1, & x < -1 \Rightarrow x \in (-\infty; -1) \end{cases}$

a) $x+1 \leq 2 \Leftrightarrow x \leq 1 \Rightarrow x \in (-\infty; 1]$

$\left.\begin{array}{l} x \in (-\infty; 1] \\ x \in [-1, \infty) \end{array}\right| \Rightarrow x \in [-1, 1]$

b) $-x-1 \leq 2 \Leftrightarrow -x \leq 3 \Leftrightarrow x \geq -3 \Rightarrow x \in [-3, \infty)$

$\left.\begin{array}{l} x \in [-3, \infty) \\ x \in (-\infty; -1) \end{array}\right| \Rightarrow$

$\Rightarrow x \in [-3; -1)$

$\left.\begin{array}{l} x \in [-1, 1] \\ x \in [-3; -1) \end{array}\right| \Rightarrow x \in [-3; 1] \Rightarrow A = \{-3, -2; -1; 0; 1\}$

Deci mulț. A are 5 elemente.

2. sunt 30 de cazuri posibile
avem cuburi perfecte $1, 2^3, 3^3$ deci avem 3 cazuri favorabile

$P = \frac{\cancel{3}}{\cancel{30}} = \frac{1}{10} = 0,1$

3. $f: \mathbb{R} \to \mathbb{R}, f(x) = x+3, g: \mathbb{R} \to \mathbb{R}, g(x) = 2x-1$

$2f(x) + 3g(x) = -5 \Leftrightarrow 2(x+3) + 3(2x-1) = -5 \Leftrightarrow 2x+6+6x-3 = -5 \Leftrightarrow$

$\Leftrightarrow 8x+3 = -5 \Leftrightarrow 8x = -8 \Rightarrow x = -1$

4. Fie x prețul înainte de reducere.

$x - x \cdot \frac{20}{100} = 320 \Leftrightarrow 100x - 20x = 32000 \Leftrightarrow 80x = 32000 \Rightarrow$

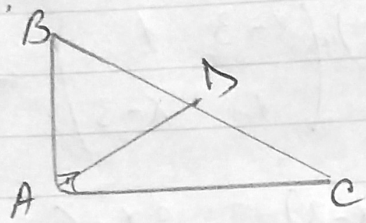
$\Rightarrow x = \frac{3200\cancel{0}}{8\cancel{0}} \Rightarrow x = 400$

5. $\vec{u} = -3\vec{i} + 2\vec{j}$ , $\vec{v} = 5\vec{i} - \vec{j}$ , $5\vec{u} + 3\vec{v} = ?$

$5\vec{u} + 3\vec{v} = 5(-3\vec{i} + 2\vec{j}) + 3(5\vec{i} - \vec{j}) = \cancel{-15\vec{i}} + 10\vec{j} +$
$+ \cancel{15\vec{i}} - 3\vec{j} = 7\vec{j}$

Deci vectorul $5\vec{u} + 3\vec{v}$ are coordonatele $(0, 7)$

6.




$\triangle dr\, ABC$
$D \in (BC)$
$BD = DC$
$AC = 6$ , $AD = 5$
$AB = ?$

Aplicăm Th. Pitagora $\Rightarrow AB^2 = BC^2 - AC^2$.
Dacă $AD = \frac{BC}{2}$ (mediana corespunzătoare ipotenuzei este jumătate din ea).

$\Rightarrow BC = 2 \cdot AD \Rightarrow BC = 2 \cdot 5 = 10$
$AB^2 = 10^2 - 6^2 = 100 - 36 = 64$
$AB = \sqrt{64} = 8$

Subiectul II

1. $A = \begin{pmatrix} x-3 & 1 \\ 1 & x-3 \end{pmatrix}$ , $x \in \mathbb{R}$ , $A^2 = A \cdot A$ , $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

a) $x = ?$ , $\det(A) = 0$.

$\det(A) = \begin{vmatrix} x-3 & 1 \\ 1 & x-3 \end{vmatrix} = (x-3)^2 - 1 = x^2 - 6x + 9 - 1 =$
$= x^2 - 6x + 8$

$\det(A) = 0 \Leftrightarrow x^2 - 6x + 8 = 0$.
$\Delta = 36 - 4 \cdot 8 = 36 - 32 = 4$

$x_{1,2} = \frac{6 \pm 2}{2}$

$\rightarrow x_1 = \frac{6+2}{2} = \frac{8}{2} = 4$

$\rightarrow x_2 = \frac{6-2}{2} = \frac{4}{2} = 2$

b) $A^2=(2x-6)A-(x^2-6x+8)\cdot I_2$

$$A^2=\begin{pmatrix} x-3 & 1 \\ 1 & x-3 \end{pmatrix}\cdot\begin{pmatrix} x-3 & 1 \\ 1 & x-3 \end{pmatrix}=\begin{pmatrix} (x-3)^2+1 & x-3+x-3 \\ x-3+x-3 & 1+(x-3)^2 \end{pmatrix}=$$

$$=\begin{pmatrix} x^2-6x+9+1 & 2x-6 \\ 2x-6 & 1+x^2-6x+9 \end{pmatrix}=\begin{pmatrix} x^2-6x+10 & 2x-6 \\ 2x-6 & x^2-6x+10 \end{pmatrix} \quad (1)$$

$$(2x-6)\cdot A-(x^2-6x+8)\cdot I_2=(2x-6)\cdot\begin{pmatrix} x-3 & 1 \\ 1 & x-3 \end{pmatrix}-$$

$$-(x^2-6x+8)\cdot\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} (2x-6)(x-3) & 3x-6 \\ 3x-6 & (2x-6)(x-3) \end{pmatrix}-$$

$$-\begin{pmatrix} x^2-6x+8 & 0 \\ 0 & x^2-6x+8 \end{pmatrix}=\begin{pmatrix} 2x^2-12x+18 & 2x-6 \\ 2x-6 & 2x^2-12x+18 \end{pmatrix}-$$

$$-\begin{pmatrix} x^2-6x+8 & 0 \\ 0 & x^2-6x+8 \end{pmatrix}=\begin{pmatrix} 2x^2-12x+18-x^2+6x-8 & 2x-6 \\ 2x-6 & 2x^2-12x+18-x^2+6x-8 \end{pmatrix}$$

$$=\begin{pmatrix} x^2-6x+10 & 2x-6 \\ 2x-6 & x^2-6x+10 \end{pmatrix} \quad (2)$$

Din (1) și (2) $\Rightarrow A^2=(2x-6)\cdot A-(x^2-6x+8)\cdot I_2$

c) $x\in\mathbb{R}$, $A^2=2\cdot A$

$$A^2=2\cdot A \Leftrightarrow \begin{pmatrix} x^2-6x+10 & 2x-6 \\ 2x-6 & x^2-6x+10 \end{pmatrix}=2\begin{pmatrix} x-3 & 1 \\ 1 & x-3 \end{pmatrix}\Leftrightarrow$$

$$\Leftrightarrow\begin{pmatrix} x^2-6x+10 & 2x-6 \\ 2x-6 & x^2-6x+10 \end{pmatrix}=\begin{pmatrix} 2x-6 & 2 \\ 2 & 2x-6 \end{pmatrix}\Rightarrow$$

$$\Rightarrow\begin{cases} x^2-6x+10=2x-6 \\ 2x-6=2 \end{cases}\Rightarrow\begin{cases} x^2-6x-2x+10+6=0 \\ 2x=2+6 \end{cases}\Leftrightarrow$$

$\Rightarrow \begin{cases} x^2-8x+16=0 \\ 2x=8 \end{cases} \Rightarrow \begin{cases} (x-4)^2=0 \\ x=8:2 \end{cases} \Rightarrow \begin{cases} (x-4)^2=0 \\ x=4 \end{cases} \Rightarrow$

$\Rightarrow x=4$ este soluție dublă

2. $x \circ y = xy - 2(x+y)+6$

a) $x \circ y = (x-2)(y-2)+2$, $(\forall)\ x,y \in \mathbb{R}$

$(x-2)(y-2)+2 = xy-2x-2y+4+2 = xy-2x-2y+6 =$
$= xy-2(x+y)+6 = x \circ y$, $\forall\ x,y \in \mathbb{R}$

b) $x \circ 2 = 2$, $\forall\ x \in \mathbb{R}$.
$x \circ 2 = (x-2)(2-2)+2 = (x-2)\cdot 0+2 = 0+2 = 2$, $\forall\ x \in \mathbb{R}$

c) $E = (-2009)\circ(-2008)\circ\ldots\circ(-1)\circ 0 \circ 1 \circ 2 \circ \ldots \circ 2009 =$

Știm din b) că $x \circ 2 = 2$, pentru $\forall\ x \in \mathbb{R}$. și știind că legea de compoziție „$\circ$" este asociativă, adică $2 \circ x = 2$, $\forall\ x \in \mathbb{R}$. $\Rightarrow E = 2$

## Subiectul III

1. $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^{2009} - 2009(x-1)-1$

a) $f(0)+f'(0) = ?$

$f'(x) = 2009 \cdot x^{2008} - 2009$
$f(0) = 0^{2009} - 2009(0-1)-1 = 0-0+2009-1 = 2008$
$f'(0) = 2009 \cdot 0^{2008} - 2009 = 0-2009 = -2009$
$f(0)+f'(0) = 2008-2009 = -1$

b) ecuatia tangentei; $\boxed{y - f(x_0) = f'(x_0)(x - x_0)}$

$A(0, 1)$

$x_0 = 0$

$y - f(0) = f'(0)(x - 0) \Leftrightarrow y - 2008 = -2009x \Rightarrow$

$\Rightarrow y + 2009x - 2008 = 0.$

c) $f$ convexă pe $[0; +\infty)$

Ca $f$ convexă trebuie să arătăm că $f''(x) \geq 0, \forall x \in [0, +\infty)$

$f'(x) = 2009 \cdot x^{2008} - 2009$

$f''(x) = 2009 \cdot 2008 \cdot x^{2007} \geq 0, \ (\forall)\ x \in [0, +\infty) \Rightarrow$

$\Rightarrow f$ este convexă pe $[0, +\infty)$

2. $f: \mathbb{R} \to \mathbb{R}, \ f(x) = x + e^{-x}$

a) $\text{Aria} = \int_0^1 f(x)\,dx = \int_0^1 (x + e^{-x})\,dx = \int_0^1 x\,dx + \int_0^1 e^{-x}\,dx =$

$= \left(\frac{x^2}{2} - e^{-x}\right)\Big|_0^1 = \frac{1}{2} - e^{-1} + 0 - e^{-0} = \overset{e}{\frac{1}{2}} - \overset{2}{\frac{1}{e}} + \overset{2e}{1} =$

$= \frac{e - 2 + 2e}{2e} = \frac{-2 + 3e}{2e}$

b) $x^2 + e^{-x^2} \geq 1, (\forall)\ x \in \mathbb{R}$ ; $\int_0^1 e^{-x^2}\,dx \geq \frac{2}{3}$

Dacă $x^2 + e^{-x^2} \geq 1$

$e^{-x^2} \geq 1 - x^2 \overset{\int_0^1}{\Longrightarrow} \int_0^1 e^{-x^2}\,dx \geq \int_0^1 (1 - x^2)\,dx \Rightarrow$

$\Rightarrow \int_0^1 e^{-x^2}\,dx \geq \int_0^1 dx - \int_0^1 x^2\,dx \Rightarrow \int_0^1 e^{-x^2}\,dx \geq \left(x - \frac{x^3}{3}\right)\Big|_0^1 \Rightarrow$

$\Rightarrow \int_0^1 e^{-x^2}\,dx \geq \overset{3}{1} - \frac{1}{3} \Rightarrow \int_0^1 e^{-x^2}\,dx \geq \frac{2}{3}$

$$V(C_g) = \pi\int_0^1 g^2(x)\,dx = \pi\int_0^1 (e^{-x}+e^{x})^2\,dx = \pi\int_0^1 (e^{-2x}+2+e^{2x})\,dx$$

$$= \pi\left(\int_0^1 e^{-2x}\,dx + \int_0^1 2\,dx + \int_0^1 e^{2x}\,dx\right) = \pi\left(-\frac{e^{-2x}}{2} + 2x + \frac{e^{2x}}{2}\right)\Bigg|_0^1 =$$

$$= \pi\left(-\frac{e^{-2}}{2} + 2 + \frac{e^{2}}{2} + \frac{e^{-2\cdot 0}}{2} - 0 - \frac{e^{2\cdot 0}}{2}\right) =$$

$$\cancel{2e^2} \quad e^2$$

$$= \pi\left(-\frac{1}{2e^2} + 2 + \frac{e^2}{2} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{2}}\right) = \pi\left(\frac{e^4+e^3+3e^2-1}{2e^2}\right)$$