Varianta 4

Subiectul I

1. $(x-1)^2+x-7<0 \Leftrightarrow x^2-2x+1+x-7<0 \Leftrightarrow x^2-x-6<0$

$x^2-x-6=0$

$\Delta=b^2-4ac$

$\Delta=(-1)^2-4\cdot1\cdot(-6)=1+24=25$

$x_{1,2}=\dfrac{-b\pm\sqrt{\Delta}}{2a}$ $\Rightarrow x_1=\dfrac{1+5}{2}=\dfrac{6}{2}=3$

$\Rightarrow x_2=\dfrac{1-5}{2}=\dfrac{-4}{2}=-2.$

x	$-\infty$		-2		3		$+\infty$
x^2-x-6	$+$	$+$	0	$-\ -$	0	$+\ +$	

$x\in(-2,3) \Rightarrow x=\{-1,0,1,2\}$

2. $a_1=1,\ a_2=3$

$a_n=a_{n-1}+r$. sau $a_n=a_1+(n-1)\cdot r$

$S_n=\dfrac{(a_1+a_n)\cdot n}{2}$

$a_2=a_1+r \Rightarrow r=a_2-a_1=3-1=2.$

pentru $n=5 \Rightarrow S_5=\dfrac{(1+a_5)\cdot5}{2}=\dfrac{(1+9)\cdot5}{2}=\dfrac{\cancel{10}^{5}\cdot5}{\cancel{2}}=25.$

$a_5=a_1+(5-1)\cdot r. \Rightarrow a_5=1+4\cdot2=9$

3. $f:\mathbb{R}\to\mathbb{R},\ f(x)=mx^2-8x-3.$

Pentru ca funcția să aibă maxim punem condiția ca $m<0 \Rightarrow m\in(-\infty,0)$

$f_{max}=-\dfrac{\Delta}{4a} \Rightarrow 5=-\dfrac{64+12m}{4m} \Leftrightarrow -64-12m=20m \Leftrightarrow$

$\Leftrightarrow 32m=-64 \Rightarrow m=-\dfrac{64}{32}=-2 \Rightarrow m=-2<0.$

4. $\log_2(x+2)-\log_2(x-5)=3$

C.E.: $\begin{cases} x+2>0 \\ x-5>0 \end{cases} \Leftrightarrow \begin{cases} x>-2 \\ x>5 \end{cases} \Rightarrow \begin{cases} x\in(-2;+\infty) \\ x\in(5,+\infty) \end{cases} \Rightarrow$

$\Rightarrow x\in(-2;+\infty)\cap(5,+\infty) \Rightarrow x\in(5,+\infty)$

$\log_2(x+2)-\log_2(x-5)=3 \Leftrightarrow \log_2\left(\frac{x+2}{x-5}\right)=\log_2 2^3 \Leftrightarrow$

$\Leftrightarrow \frac{x+2}{x-5}=8 \Leftrightarrow x+2=8(x-5) \Leftrightarrow x+2-8x=-40 \Leftrightarrow$

$x\neq 5$

$\Leftrightarrow -7x=-42 \Rightarrow x=\frac{-42}{-7} \Rightarrow x=6\in(5,+\infty)$

5. $a=?$, $\vec{u}=2\vec{i}+a\vec{j}$, $\vec{v}=3\vec{i}+(a-2)\vec{j}$

$\vec{u}$ și $\vec{v}$ sunt coliniari $\Leftrightarrow \frac{2}{3}=\frac{a}{a-2} \Rightarrow$

$\Rightarrow 2(a-2)=3a \Leftrightarrow 2a-4=3a \Leftrightarrow 3a-2a=-4 \Rightarrow$

$\Rightarrow a=-4$

6. $\triangle ABC$, $AB=3$, $m(\hat{C})=30^\circ$, $R=?$

Aplicăm Th. sinusului: $\boxed{\frac{c}{\sin\hat{C}}=2R}$

$AB=c$

$\frac{AB}{\sin\hat{C}}=2R \Rightarrow R=\frac{AB}{2\cdot\sin\hat{C}}=\frac{3}{2\cdot\sin 30^\circ}=\frac{3}{\cancel{2}\cdot\frac{1}{\cancel{2}}}=3.$

Subiectul II.

1. $I_2=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, $A=\begin{pmatrix}4&-6\\2&-3\end{pmatrix}$, $X(a)=I_2+aA$, $a\in\mathbb{R}$

a) $A^3=?$

$$A^3=A\cdot A\cdot A=\begin{pmatrix}4&-6\\2&-3\end{pmatrix}\begin{pmatrix}4&-6\\2&-3\end{pmatrix}\begin{pmatrix}4&-6\\2&-3\end{pmatrix}=$$

$$=\begin{pmatrix}16-12&-24+18\\8-6&-12+9\end{pmatrix}\begin{pmatrix}4&-6\\2&-3\end{pmatrix}=\begin{pmatrix}4&-6\\2&-3\end{pmatrix}\begin{pmatrix}4&-6\\2&-3\end{pmatrix}=\begin{pmatrix}4&-6\\2&-3\end{pmatrix}$$

b) $X(a)\cdot X(b)=X(a+b+ab)$, $(\forall)\ a,b\in\mathbb{R}$

$$\left.\begin{array}{l}X(a)=I_2+aA\\X(b)=I_2+bA\end{array}\right\} \Rightarrow X(a)\cdot X(b)=(I_2+aA)\cdot(I_2+bA)=$$

$$=I_2^2+aAI_2+bA\cdot I_2+(a\cdot b)\cdot A^2=$$

$$=I_2+aA+bA+a\cdot b\cdot A=I_2+(a+b+a\cdot b)A$$

$$X(a+b+ab)=I_2+(a+b+ab)A \quad \Rightarrow$$

$$\Rightarrow X(a)\cdot X(b)=X(a+b+ab)$$

c) $X(1)+X(2)+X(3)+\dots+X(2009)=\underbrace{I_2+A}_{X(1)}+\underbrace{I_2+2A}_{X(2)}+\dots+$

$+\underbrace{I_2+2009\cdot A}_{X(2009)}=2009\cdot I_2+(1+2+3+\dots+2009)\cdot A=$

ne apare I_2 de 2009 ori

$$=2009\,I_2+\frac{2009(2009+1)}{2}\cdot A=2009\,I_2+\frac{2009\cdot \cancel{2010}^{1005}}{\cancel{2}}A=$$

$$\boxed{1+2+3+\dots+n=\frac{n(n+1)}{2}}$$

$$=2009\,I_2+2009\cdot 1005\,A=2009(I_2+1005A)$$

2. $(\mathbb{Z}_6,+,\cdot)$, $\mathbb{Z}_6=\{\hat{0},\hat{1},\hat{2},\hat{3},\hat{4},\hat{5}\}$

a) $\hat{2}x+\hat{5}=\hat{1} \Leftrightarrow \hat{2}x=\hat{1}-\hat{5} \Leftrightarrow \hat{2}x=-\hat{4} \Leftrightarrow$

$\Leftrightarrow \hat{2}x=\hat{6}-\hat{4} \Leftrightarrow \hat{2}x=\hat{2} \Rightarrow x=\{\hat{1},\hat{4}\}$

x	$\hat{0}$	$\hat{1}$	$\hat{2}$	$\hat{3}$	$\hat{4}$	$\hat{5}$
$\hat{2}x$	$\hat{0}$	$\underline{\hat{2}}$	$\hat{4}$	$\hat{0}$	$\underline{\hat{2}}$	$\hat{4}$

b) $\left|\begin{array}{ccc} \hat{1} & \hat{2} & \hat{3} \\ \hat{2} & \hat{3} & \hat{1} \\ \hat{3} & \hat{1} & \hat{2} \\ \hat{1} & \hat{2} & \hat{3} \\ \hat{2} & \hat{3} & \hat{1} \end{array}\right| = \hat{1}\cdot\hat{3}\cdot\hat{2}+\hat{2}\cdot\hat{1}\cdot\hat{3}+\hat{3}\cdot\hat{2}\cdot\hat{1}-\hat{3}\cdot\hat{3}\cdot\hat{3}-\hat{1}\cdot\hat{1}\cdot\hat{1}-$

$-\hat{2}\cdot\hat{2}\cdot\hat{2}=(\hat{6}+\hat{6}+\hat{6}-\hat{27}-\hat{1}-\hat{8})=$

$=\hat{0}-\hat{3}-\hat{1}-\hat{2}=\hat{0}+\hat{3}+\hat{5}+\hat{4}=(\hat{0}+\hat{12})=\hat{0}$

c) $\begin{cases} \hat{2}x+y=\hat{4} \\ x+\hat{2}y=\hat{5} \quad |\cdot\hat{4} \end{cases} \Leftrightarrow \begin{cases} \hat{2}x+y=\hat{4} \\ \hat{4}x+\hat{2}y=\hat{2} \end{cases} \Rightarrow$

$\hat{6}x+\hat{3}y=\hat{6} \Rightarrow \hat{0}x+\hat{3}y=\hat{0} \Rightarrow$

$\Rightarrow \hat{3}y=\hat{0} \Rightarrow y=\{\hat{0},\hat{2},\hat{4}\}$

y	$\hat{0}$	$\hat{1}$	$\hat{2}$	$\hat{3}$	$\hat{4}$	$\hat{5}$
$\hat{3}y$	$\underline{\hat{0}}$	$\hat{3}$	$\underline{\hat{0}}$	$\hat{3}$	$\underline{\hat{0}}$	$\hat{3}$

Cazul I $\begin{cases} y=\hat{0} \\ x+\hat{2}y=\hat{5} \end{cases} \Leftrightarrow \begin{cases} y=\hat{0} \\ x=\hat{5}-\hat{0}=\hat{5} \end{cases} \Rightarrow \begin{cases} y=\hat{0} \\ x=\hat{5} \end{cases}$

Cazul II $\begin{cases} y=\hat{2} \\ x+\hat{2}y=\hat{5} \end{cases} \Leftrightarrow \begin{cases} y=\hat{2} \\ x=\hat{5}-\hat{4} \end{cases} \Leftrightarrow \begin{cases} y=\hat{2} \\ x=\hat{5}+\hat{2} \end{cases} \Rightarrow \begin{cases} y=\hat{2} \\ x=\hat{1} \end{cases}$

Cazul III $\begin{cases} y=\hat{4} \\ x+\hat{2}y=\hat{5} \end{cases} \Leftrightarrow \begin{cases} y=\hat{4} \\ x=\hat{5}-\hat{2} \end{cases} \Leftrightarrow \begin{cases} y=\hat{4} \\ x=\hat{5}+\hat{4} \end{cases} \Rightarrow \begin{cases} y=\hat{4} \\ x=\hat{3} \end{cases}$

(Mai sunt și alte metode de a rezolva sistemul)

Subiectul III

1. $f:\mathbb{R}\to\mathbb{R},\ f(x)=x+e^{-x}$

a) $f'(x)=(x+e^{-x})'=1-e^{-x}$

b)

x	$-\infty$		0		$+\infty$
$f'(x)$	$-\ -\ -\ -\ -$		0	$+\ +\ +\ +$	
$f(x)$	$\searrow$		1	$\nearrow$	

$f'(x)=0 \Rightarrow 1-e^{-x}=0 \Leftrightarrow -e^{-x}=-1 \Leftrightarrow e^{-x}=1 \Leftrightarrow$

$\Leftrightarrow e^{-x}=e^{0} \Rightarrow -x=0 \Rightarrow x=0.$

$f'(0)=1-e^{-x}=0.$

$f(0)=0+e^{-0}=1+0=1$

Pe intervalul $(-\infty,0]$ funcția este descrescătoare
Pe intervalul $[0,+\infty)$ funcția este crescătoare

c) Ecuația asimtotei oblice : $y=mx+n$

$$m=\lim_{x\to+\infty}\frac{f(x)}{x}=\lim_{x\to+\infty}\frac{x+e^{-x}}{x}=\lim_{x\to+\infty}\left(1+\underbrace{\frac{e^{-x}}{x}}_{0}\right)=1.$$

$$n=\lim_{x\to+\infty}(f(x)-mx)=\lim_{x\to+\infty}(\cancel{x}+e^{-x}-\cancel{x})=\lim_{x\to+\infty}e^{-x}=0$$

$y=x+0 \Rightarrow y=x$

2. $g:\mathbb{R}\to\mathbb{R},\ g(x)=(x+1)^3-3x^2-1$

a) $$\int_0^1 g(x)\,dx=\int_0^1\left[(x+1)^3-3x^2-1\right]dx=\int_0^1(x^3+\cancel{3x^2}+3x+\cancel{1}-\cancel{3x^2}-\cancel{1})\,dx$$

$$=\int_0^1(x^3+3x)\,dx=\int_0^1 x^3\,dx+\int_0^1 3x\,dx=\frac{x^4}{4}\Big|_0^1+3\frac{x^2}{2}\Big|_0^1=$$

$$=\frac{1}{4}+3\cdot\frac{1}{2}=\frac{1}{4}+\frac{3}{2}=\frac{7}{4}$$

b) $a>1$, $\int_1^a (g(x)-x^3)\cdot e^x\,dx = 6e^a$

$$\int_1^a \left[(x+1)^3-3x^2-1-x^3\right]\cdot e^x\,dx = 6e^a$$

$$\int_1^a (\cancel{x^3}+\cancel{3x^2}+3x+\cancel{1}-\cancel{3x^2}-\cancel{1}-\cancel{x^3})\cdot e^x\,dx = 6e^a$$

$$\int_1^a 3x\cdot e^x\,dx = 6e^a$$

$$\int_1^a 3x\cdot e^x\,dx = 3\int_1^a x\cdot e^x\,dx = 3\left[x\cdot e^x\Big|_1^a-\int_1^a e^x\,dx\right] =$$

$f(x)=x \quad f'(x)=1$

$g'(x)=e^x \quad g(x)=e^x$

$$= 3\left(ae^a-e-e^x\Big|_1^a\right) = 3\left(ae^a-\cancel{e}-e^a+\cancel{e}\right) = 3\left(ae^a-e^a\right)$$

$$3(ae^a-e^a) = 6e^a \Leftrightarrow 3e^a(a-1) = 6e^a \Leftrightarrow$$

$$\Leftrightarrow a-1 = \frac{\overset{2}{\cancel{6e^a}}}{\cancel{3e^a}} \Leftrightarrow a-1 = 2 \Rightarrow a = 3$$

c) $\int_0^1 (3x^2+3)\cdot g^{2009}(x)\,dx = \int_0^1 (3x^2+3)(x^3+3x)^{2009}\,dx =$

$$g(x) = (x+1)^3-3x^2-1 = x^3+\cancel{3x^2}+3x+\cancel{1}-\cancel{3x^2}-\cancel{1} = x^3+3x$$

$u = x^3+3x$

$du = u' = (3x^2+3)\,dx$

$$= \int_0^1 u^{2009}\,du = \frac{u^{2010}}{2010}\Big|_0^1 = \frac{(x^3+3x)^{2010}}{2010}\Big|_0^1 =$$

$$= \frac{(1+3)^{2010}}{2010} - 0 = \frac{4^{2010}}{2010}.$$

sau

$$\int_0^1 (3x^2+3)\cdot g^{2009}(x)\,dx = \int_0^1 (3x^2+3)(x^3+3x)^{2009}\,dx =$$

$$= \int_0^1 (3x^2+3)'(x^3+3x)^{2009}\,dx = \frac{(x^3+3x)^{2010}}{2010}\Big|_0^1 = \frac{4^{2010}}{2010}.$$