Model

2015 – M_tehnologic

Subiectul I

1. $a = 2(5-\sqrt{5})$

$b = 2\sqrt{5}$

$m_a = \frac{a+b}{2} = \frac{2(5-\sqrt{5})+2\sqrt{5}}{2} = \frac{\cancel{2}(5-\cancel{\sqrt{5}}+\cancel{\sqrt{5}})}{\cancel{2}} = 5$

2. $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 - 4x + 3$.

$G_f \cap Ox \Rightarrow y = 0 \Rightarrow f(x) = 0$

$x^2 - 4x + 3 = 0 \Leftrightarrow x^2 - x - 3x + 3 = 0 \Leftrightarrow$

$\Leftrightarrow x(x-1) - 3(x-1) = 0 \Leftrightarrow (x-1)(x-3) = 0 \Rightarrow$

$\Rightarrow x - 1 = 0 \Rightarrow x_1 = 1$

$\Rightarrow x - 3 = 0 \Rightarrow x_1 = 3$.

3. $\log_5(2x-1) - \log_5 3 = 0 \Leftrightarrow \log_5(2x-1) = \log_5 3 \Rightarrow$

C.E. $2x-1 > 0 \Rightarrow x > \frac{1}{2}$

$\Rightarrow 2x - 1 = 3 \Rightarrow 2x = 3+1 \Rightarrow 2x = 4 \Rightarrow x = 4 : 2 \Rightarrow x = 2$.

4. $P = \frac{\text{nr. cazurilor favorabile}}{\text{nr. cazurilor posibile}} = \frac{4^{\backslash 2}}{10} = \frac{2}{5}$

$M_3 = \{0, 3, 6, 9\}$

nr. cazurilor favorabile $= 4$

numerele de o cifră sunt: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 $\Rightarrow$

$\Rightarrow$ nr. cazurilor posibile $= 10$.

5. $A(2,4)$, $B(6,4)$

Fie $C(x_C, y_C)$ mijlocul segmentului AB. $\Rightarrow$

$$\Rightarrow C(x_C, y_C) = C\left(\frac{x_A+x_B}{2}; \frac{y_A+y_B}{2}\right)$$

$$C(x_C, y_C) = C\left(\frac{2+6}{2}; \frac{4+4}{2}\right) = C(4,4) \Rightarrow$$

$\Rightarrow x_C = 4$, $y_C = 4$

Deci, coordonatele mijlocului segmentului AB sunt $x_C = 4$ și $y_C = 4$.

6. $\sin(a+b) = \frac{63}{65}$, $a, b \in \left(0, \frac{\pi}{2}\right)$, $\sin a = \frac{3}{5}$, $\sin b = \frac{12}{13}$.

$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$

$$\sin^2 a + \cos^2 a = 1 \Rightarrow \cos^2 a = 1 - \sin^2 a \Rightarrow$$

$$\Rightarrow \cos^2 a = 1 - \left(\frac{3}{5}\right)^2 = \overset{25)}{1} - \frac{9}{25} = \frac{16}{25}$$

$$a \in \left(0, \frac{\pi}{2}\right) \Bigg/ \Rightarrow \cos a = \frac{4}{5}$$

$$\cos^2 b = 1 - \sin^2 b \Rightarrow \cos^2 b = 1 - \left(\frac{12}{13}\right)^2 = \overset{169)}{1} - \frac{144}{169} = \frac{25}{169} \Rightarrow$$

$$\Rightarrow \cos^2 b = \frac{25}{169}, \quad b \in \left(0, \frac{\pi}{2}\right) \Rightarrow \cos b = \frac{5}{13}.$$

$$\sin(a+b) = \frac{3}{5} \cdot \frac{5}{13} + \frac{4}{5} \cdot \frac{12}{13} = \frac{15+48}{65} = \frac{63}{65}$$

Subiectul al II-lea

1. a) $\det A = \begin{vmatrix} 2 & -2 \\ 1 & -1 \end{vmatrix} = -2+2=0.$

b) $p = ?$

$A \cdot A = p \cdot A \Leftrightarrow \begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix} = p\begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix} \Leftrightarrow$

$\Leftrightarrow \begin{pmatrix} 4-2 & -4+2 \\ 2-1 & -2+1 \end{pmatrix} = p\begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix} = p\begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix} \Rightarrow$

$\Rightarrow p = 1.$

Sau.

$\begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix} = p\begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 2p & -2p \\ p & -p \end{pmatrix} \Rightarrow$

$\Rightarrow 2p = 2 \Rightarrow p = 2:2 = 1 \Rightarrow p = 1.$

c) $B = \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}$, $\det(A+B)=0$, $b \in \mathbb{R}$

$A+B = \begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix} + \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix} = \begin{pmatrix} 2 & -2+b \\ 1+b & -1 \end{pmatrix}$

$\det(A+B) = \begin{vmatrix} 2 & -2+b \\ 1+b & -1 \end{vmatrix} = -2-(-2+b)(1+b) =$

$= -2-(-2-2b+b+b^2) =$

$= -\cancel{2}+\cancel{2}+b-b^2 = b-b^2$

$\det(A+B)=0 \Leftrightarrow b-b^2=0 \Leftrightarrow b(1-b)=0 \Rightarrow$

$\Rightarrow b=0$. sau $1-b=0 \Rightarrow b=1.$

Pentru $b=0 \Rightarrow B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Pentru $b=1 \Rightarrow B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

2. $x \circ y = -xy + x + y$

a) $1 \circ 2015 = -1 \cdot 2015 + 1 + 2015 = -2015 + 2016 = 1$.

b) $x \circ y = -(x-1)(y-1) + 1$, $\forall\, x, y \in \mathbb{R}$.

$-(x-1)(y-1) + 1 = -(xy - x - y + 1) + 1 = -xy + x + y \cancel{-1+1} =$

$= -xy + x + y$

$x \circ y = -xy + x + y$ $\Big/ \Rightarrow x \circ y = -(x-1)(y-1) + 1$

c) $3^x \circ 5^x = 1 \Leftrightarrow -3^x \cdot 5^x + 3^x + 5^x = 1 \Leftrightarrow$

$\Leftrightarrow -(3^x - 1)(5^x - 1) + 1 = 1 \Leftrightarrow -(3^x - 1)(5^x - 1) = 1 - 1 \Leftrightarrow$

$\Leftrightarrow -(3^x - 1)(5^x - 1) = 0 \overset{\cdot(-1)}{\Leftrightarrow} (3^x - 1)(5^x - 1) = 0 \Rightarrow$

$\Rightarrow 3^x - 1 = 0 \Rightarrow 3^x = 1 \Rightarrow 3^x = 3^0 \Rightarrow x = 0$.

$5^x - 1 = 0 \Rightarrow 5^x = 1 \Rightarrow 5^x = 5^0 \Rightarrow x = 0$.

Deci, soluția finală $x = 0$.

Subiectul al III-lea.

1. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \dfrac{3x}{x^2+1}$

a) $\lim\limits_{x \to 1} f(x) = \lim\limits_{x \to 1} \dfrac{3x}{x^2+1} = \dfrac{3 \cdot 1}{1^2+1} = \dfrac{3}{2}$.

b) $f'(x) = \left(\dfrac{3x}{x^2+1}\right)' = \dfrac{(3x)' \cdot (x^2+1) - 3x \cdot (x^2+1)'}{(x^2+1)^2} =$

$= \dfrac{3(x^2+1) - 3x \cdot 2x}{(x^2+1)^2} = \dfrac{3x^2 + 3 - 6x^2}{(x^2+1)^2} = \dfrac{3 - 3x^2}{(x^2+1)^2} =$

$= -\dfrac{3x^2 - 3}{(x^2+1)^2} = -\dfrac{3(x^2-1)}{(x^2+1)^2} = -\dfrac{3(x-1)(x+1)}{(x^2+1)^2}$

c) În primul rând facem tabelul de semne pentru $f'(x)$

x	$-\infty$		-1		1		$+\infty$
$-3(x-1)(x+1)$	$- \; - \; -$		0	$+ \; + \; +$	0	$- \; - \; -$	
$(x^2+1)^2$	$+ \; + \; +$		$+$	$+ \; + \; +$	$+$	$+ \; + \; +$	
$f'(x)$	$- \; - \; -$		0	$+ \; +$	0	$- \; - \; -$	

$\cdot \; x-1=0 \Rightarrow x=1$

$x+1=0 \Rightarrow x=-1$

$-3(x-1)(x+1) = -3x^2+3 \Rightarrow$ avem semn contrar între rădăcini și semnul lui a în afara lor, unde $a=-3$.

$(x^2+1)^2 > 0 \Rightarrow$ semnul este pozitiv $\forall x \in \mathbb{R}$.

$$f'(-1) = \frac{-3(-1-1)(-1+1)}{((-1)^2+1)^2} = \frac{0}{2^2} = \frac{0}{4} = 0.$$

$$f'(1) = \frac{-3(1-1)(1+1)}{(1+1)^2} = \frac{0}{4} = 0.$$

Acum facem tabelul de variație:

x	$-\infty$		-1		1		$+\infty$
$f(x)$		$\searrow$	$-\frac{3}{2}$	$\nearrow$	$\frac{3}{2}$	$\searrow$	
$f'(x)$	$- \; - \; -$		0	$+ \; + \; + \; +$	0	$- \; - \; - \; -$	

$f(-1) = \frac{-3}{(-1)^2+1} = -\frac{3}{2}$ $\qquad f(1) = \frac{3}{1+1} = \frac{3}{2}$

$f'(x) \leq 0, \forall x \in (-\infty; -1] \Rightarrow f$ este descrescătoare pe $(-\infty; -1]$

$f'(x) \geq 0, \forall x \in [-1, 1] \Rightarrow f$ este crescătoare pe $[-1, 1]$

$f'(x) \leq 0, \forall x \in [1, +\infty) \Rightarrow f$ este descrescătoare pe $[1, +\infty)$

2. $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^5 + x$

a) $\int_{-1}^{1} x^5 dx = \frac{x^6}{6}\Big|_{-1}^{1} = \frac{1^6}{6} - \frac{(-1)^6}{6} = \frac{1}{6} - \frac{1}{6} = 0$

b) $\int_0^1 (f(x) - x^5) e^x dx = \int_0^1 (\cancel{x^5} + x - \cancel{x^5}) e^x dx =$

$= \int_0^1 x e^x dx = (x e^x)\Big|_0^1 - \int_0^1 e^x dx = (x e^x)\Big|_0^1 - e^x\Big|_0^1 =$

$f(x) = x \quad f'(x) = 1$

$g'(x) = e^x \quad g(x) = e^x$

$= (x e^x - e^x)\Big|_0^1 = \cancel{e^1} - \cancel{e^1} - 0 + e^0 = 1 \Rightarrow$

$\Rightarrow \int_0^1 (f(x) - x^5) e^x dx = 1$

c) $V = \pi \int_1^2 (g(x))^2 dx$

$g(x) = \frac{f(x) - x}{x^3} = \frac{x^5 + \cancel{x} - \cancel{x}}{x^3} = \frac{x^5}{x^3} = x^2$

$V = \pi \int_1^2 (x^2)^2 dx = \pi \frac{x^5}{5}\Big|_1^2 = \pi\left(\frac{2^5}{5} - \frac{1}{5}\right) = \pi \cdot \frac{32-1}{5} =$

$= \pi \cdot \frac{31}{5} = \frac{31}{5}\pi$