Simulare
2016 – M_ tehnologic

Subiectul I

1. $a, b = ?$

$\frac{10}{3+i} = a+ib \Rightarrow \frac{10(3-i)}{(3+i)(3-i)} = a+ib \Rightarrow$

$\Rightarrow \frac{30-10i}{9-i^2} = a+ib \Rightarrow \frac{30-10i}{9+1} = a+ib \Rightarrow$

$i^2 = -1$

$\Rightarrow \frac{30-10i}{10} = a+ib \Rightarrow \frac{30}{10} - \frac{10}{10}i = a+ib \Rightarrow$

$\Rightarrow 3 - i = a+ib \Rightarrow a = 3$

$b = -1$.

2. $f:\mathbb{R}\to\mathbb{R},\ f(x) = x^2 - 1$

$(f(1))^{2016} + (f(0))^{2016} = 0^{2016} + (-1)^{2016} = 0 + 1 = 1$.

$f(1) = 1^2 - 1 = 1 - 1 = 0$.

$f(0) = 0 - 1 = -1$.

3. $6^{x^2-3x+5} = 216 \Leftrightarrow 6^{x^2-3x+5} = 6^3 \Rightarrow x^2-3x+5 = 3 \Leftrightarrow$

$216 = 6^3$

$\Leftrightarrow x^2 - 3x + 5 - 3 = 0 \Leftrightarrow x^2 - 3x + 2 = 0 \Leftrightarrow x^2 - x - 2x + 2 = 0 \Leftrightarrow$

$\Leftrightarrow x(x-1) - 2(x-1) = 0 \Leftrightarrow (x-1)(x-2) = 0 \Rightarrow$

$\Rightarrow x - 1 = 0 \Rightarrow x_1 = 1$.

sau

$x - 2 = 0 \Rightarrow x_2 = 2$.

4. $C_6^5 = \frac{6!}{5!(6-5)!} = \frac{6}{1!} = \frac{6}{1} = 6$ moduri

$C_n^k = \frac{n!}{k!(n-k)!}$

5. $A(5,0)$, $B(2m+1, 0)$, $C(10,0)$.

C este mijlocul lui AB:

$C(x_C, y_C) = C\left(\frac{x_A + x_B}{2}, \frac{y_A + y_B}{2}\right)$

$C(10,0) = C\left(\frac{5+2m+1}{2}, \frac{0+0}{2}\right)$

$C(10,0) = C\left(\frac{5+2m+1}{2}, 0\right) \Rightarrow$

$\Rightarrow \frac{6+2m}{2} = 10 \Rightarrow \frac{\cancel{2}(3+m)}{\cancel{2}} = 10 \Rightarrow 3 + m = 10 \Rightarrow$

$\Rightarrow m = 10 - 3 \Rightarrow m = 7$

6. $\triangle ABC$, $AB = 5$, $AC = 12$, $BC = 13$

Observăm că $BC^2 = AB^2 + AC^2 \Rightarrow$

$(13^2 = 5^2 + 12^2)$

$\Rightarrow \triangle ABC$ este $\triangle$ dreptunghic în A.

Deci, $\cos C = \frac{AC}{BC} = \frac{12}{13}$.

Subiectul al II-lea

1. a) $\det A = \begin{vmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{vmatrix} = 1 + 0 + 0 - 0 - 0 - 0 = 1$

b) $(A - I_3)(A - I_3)(A - I_3) = O_3$?

$A - I_3 = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 4 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}$

$$(A-I_3)(A-I_3)(A-I_3)=\begin{pmatrix}0&2&4\\0&0&3\\0&0&0\end{pmatrix}\begin{pmatrix}0&2&4\\0&0&3\\0&0&0\end{pmatrix}\begin{pmatrix}0&2&4\\0&0&3\\0&0&0\end{pmatrix}=$$

$$=\begin{pmatrix}0&0&6\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&2&4\\0&0&3\\0&0&0\end{pmatrix}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}=O_3 \Rightarrow$$

$$\Rightarrow (A-I_3)(A-I_3)(A-I_3)=O_3$$

c) $AX=\begin{pmatrix}0\\1\\2\end{pmatrix}$, $X=\begin{pmatrix}x\\y\\z\end{pmatrix}\in \mathcal{M}_{3,1}(\mathbb{R})$

$$AX=\begin{pmatrix}1&2&4\\0&1&3\\0&0&1\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}x+2y+4z\\y+3z\\z\end{pmatrix}$$

$$AX=\begin{pmatrix}0\\1\\2\end{pmatrix} \Leftrightarrow \begin{pmatrix}x+2y+4z\\y+3z\\z\end{pmatrix}=\begin{pmatrix}0\\1\\2\end{pmatrix} \Rightarrow$$

$$\Rightarrow \begin{cases}z=2\\y+3z=1\\x+2y+4z=0\end{cases} \Leftrightarrow \begin{cases}z=2\\y=1-3z\\x=-2y-4z\end{cases} \Leftrightarrow \begin{cases}z=2\\y=1-3\cdot 2\\x=-2y-8\end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases}z=2\\y=-5\\x=10-8\end{cases} \Rightarrow \begin{cases}x=2\\y=-5\\z=2.\end{cases}$$

2. $x*y=xy-x-y+2$

a) $x*y=(x-1)(y-1)+1$, $\forall x,y\in\mathbb{R}$.

$$\left.\begin{array}{l}(x-1)(y-1)+1=xy-x-y+1+1=xy-x-y+2\\ x*y=xy-x-y+2\end{array}\right\} \Rightarrow$$

$$\Rightarrow x*y=(x-1)(y-1)+1,\ \forall x,y\in\mathbb{R}$$

b) $0*1*2*3=(0*1)*(2*3)=1*3=\cancel{1\cdot 3-1-3+2}=1$.

$0*1=0\cdot 1-0-1+2=1$

$2*3=2\cdot 3-\cancel{2}-3+\cancel{2}=6-3=3.$

c) $a = ?$

$a * a * 2016 = 2016$.

$a * a = a \cdot a - a - a + 2 = a^2 - 2a + 2 = (a-1)^2 + 1$

$a * a * 2016 = (a * a) * 2016 = [(a-1)^2 + 1] * 2016 =$

$= [(a-1)^2 + \cancel{1} - \cancel{1}][2016 - 1] + 1 = 2015(a-1)^2 + 1$

stim că $x * y = (x-1)(y-1) + 1$.

Deci:

$a * a * 2016 = 2016 \Leftrightarrow 2015(a-1)^2 + 1 = 2016 \Leftrightarrow$

$\Leftrightarrow 2015(a-1)^2 = 2016 - 1 \Leftrightarrow 2015(a-1)^2 = 2015 \;/:2015 \Leftrightarrow$

$\Leftrightarrow (a-1)^2 = 1 \Leftrightarrow (a-1)^2 - 1 = 0 \Leftrightarrow (a-1-1)(a-\cancel{1}+\cancel{1}) = 0 \Rightarrow$

aplicăm $a^2 - b^2 = (a-b)(a+b)$

$\Rightarrow (a-2) \cdot a = 0 \Rightarrow a = 0$

sau

$a - 2 = 0 \Rightarrow a = 2$

Subiectul al III-lea

1. $f : (0, +\infty) \to \mathbb{R}$, $f(x) = \frac{x+1}{x}$

a) $\lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = f'(2)$

$f'(x) = \left(\frac{x+1}{x}\right)' = \frac{(x+1)' \cdot x - (x+1) \cdot (x)'}{x^2} = \frac{x - (x+1)}{x^2} = \frac{\cancel{x} - \cancel{x} - 1}{x^2} = \frac{-1}{x^2}$

$f'(2) = \frac{-1}{2^2} = -\frac{1}{4}$

$\lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = -\frac{1}{4}$

b) ec tangentei la graficul funcției f în punctul $x=1$

este: $y - f(1) = f'(1)(x-1)$

$f(1) = \frac{1+1}{1} = 2$

$f'(1) = -\frac{1}{1} = -1$

$y - f(1) = f'(1)(x-1) \Leftrightarrow y - 2 = -1(x-1) \Leftrightarrow$

$\Leftrightarrow y = -x+1+2 \Rightarrow y = -x+3$.

c) $\frac{2017}{2016} \leq f(x) \leq 2$, $\forall x \in [1, 2016]$

$f(x) = \frac{x+1}{x}$

$f'(x) = \frac{(x+1)' \cdot x - (x+1) \cdot x'}{x^2} = \frac{x-(x+1)}{x^2} = \frac{\cancel{x} - \cancel{x} - 1}{x^2} = -\frac{1}{x^2}$

x	0		1		2016		$+\infty$
$f'(x)$	$\vert$	$-\ -$	-1	$-$	$-\frac{1}{2016^2}$	$-$	$-$
$f(x)$	$\vert$	$\searrow$	2	$\searrow$	$\frac{2017}{2016}$	$\searrow$	

$f'(0) = -\frac{1}{0}$ nu se poate

$f(1) = \frac{1+1}{1} = 2$.

$f(2016) = \frac{2016+1}{2016} = \frac{2017}{2016}$

$f(0) = \frac{1}{0}$ nu se poate

$f'(x) < 0$, $(\forall) x \in (0, +\infty) \Rightarrow f$ descrescătoare pe $(0, +\infty)$.

Cum $[1, 2016] \in (0, +\infty)$

$\Rightarrow \frac{2017}{2016} \leq f(x) \leq 2$, $(\forall)\ x \in [1, 2016]$

2. $f:\mathbb{R}\to\mathbb{R}$, $f(x)=x^3-3x^2+2$

a) $\int_0^2 (f(x)+3x^2-2)\,dx = \int_0^2 (x^3-\cancel{3x^2}+\cancel{2}+\cancel{3x^2}-\cancel{2})\,dx =$

$=\int_0^2 x^3\,dx = \frac{x^4}{4}\Big|_0^2 = \frac{2^4}{4}-0=\frac{16}{4}=4$

b) $\int_0^1 (f(x)-x^3+3x^2+x)e^x\,dx = \int_0^1(\cancel{x^3}-\cancel{3x^2}+2-\cancel{x^3}+\cancel{3x^2}+x)e^x\,dx=$

$=\int_0^1(2+x)e^x\,dx = (2+x)e^x\Big|_0^1 - \int_0^1 e^x\,dx =$

$f(x)=2+x \quad f'(x)=1$

$g'(x)=e^x \quad g(x)=e^x$

$=\left[(2+x)e^x - e^x\right]\Big|_0^1 = \left[e^x(2+x-1)\right]\Big|_0^1 =$

$=\left[e^x(x+1)\right]\Big|_0^1 = e^1(1+1)-e^0(0+1) = 2e-1$

Deci

$\int_0^1 (f(x)-x^3+3x^2+x)e^x\,dx = 2e-1$

c) $\int_{1-a}^{1+a} f(x)\,dx = \int_{1-a}^{1+a}(x^3-3x^2+2)\,dx = \left(\frac{x^4}{4}-\frac{\cancel{3}\cdot x^3}{\cancel{3}}+2x\right)\Big|_{1-a}^{1+a}=$

$=\frac{(1+a)^4}{4}-(1+a)^3+2(1+a)-\left[\frac{(1-a)^4}{4}-(1-a)^3+2(1-a)\right]=$

$=\frac{a^4+4a^3+6a^2+4a+1}{4}-(a^3+3a^2+3a+1)+\cancel{2}+2a-\frac{a^4-4a^3+6a^2-4a+1}{4}+$

$+1-3a+3a^2-a^3-\cancel{2}+2a = \cancel{\frac{a^4}{4}}+\cancel{a^3}+\cancel{\frac{6a^2}{4}}+a+\cancel{\frac{1}{4}}-\cancel{\frac{a^4}{4}}+a^3-$

$-\cancel{\frac{6a^2}{4}}+a-\cancel{\frac{1}{4}}-\cancel{a^3}-\cancel{3a^2}-3a-\cancel{1}+2a+\cancel{1}-3a+\cancel{3a^2}-\cancel{a^3}+2a=$

$=2a-3a+2a-3a+2a=6a-6a=0,\ \forall a\in\mathbb{R}$.