Varianta 1 (2015 M. tehnologic)

Subiectul I

1. $\frac{1}{2} : 0,5 - 1 = 0,5 : 0,5 - 1 = 1 - 1 = 0 \Rightarrow \frac{1}{2} : 0,5 - 1 = 0.$

2. $f(-1) + f(0) + f(1) = ?$ $f(-1)+f(0)+f(1) = 0+0+2 = 2$

$f : \mathbb{R} \to \mathbb{R}, f(x) = x^2 + x.$

$f(-1) = (-1)^2 - 1 = 1 - 1 = 0.$

$f(0) = 0 + 0 = 0$

$f(1) = 1^2 + 1 = 1 + 1 = 2$

3. $\sqrt{3x+1} = 5 \overset{()^2}{\Leftrightarrow} 3x + 1 = 25 \Leftrightarrow 3x = 24 \Rightarrow x = 24 : 3 \Rightarrow x = 8$

C.E. $3x + 1 > 0 \Rightarrow 3x > 0 - 1 \Rightarrow x > -\frac{1}{3}.$

Soluția finală: $x = 8$, care verifică ecuația.

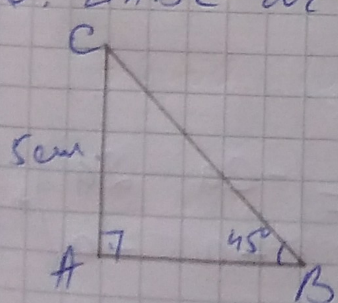
4. $150 + \frac{3\cancel{0}}{10\cancel{0}} \cdot 15\cancel{0} = 150 + 45 = 195$ lei

5. $A(1,5)$, $B(3,5)$

$d(A,B) = AB.$

$AB = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2} = \sqrt{(3-1)^2 + (5-5)^2} = \sqrt{2^2} = 2$

6. ΔABC dr. în $\measuredangle A$, $AC = 5$, $m(\measuredangle B) = 45^\circ.$



$m(\measuredangle C) = 180^\circ - m(\measuredangle A) - m(\hat{B}) =$

$= 180^\circ - 90^\circ - 45^\circ = 45^\circ \Rightarrow$

$\Rightarrow m(\measuredangle C) = 45^\circ$

$m(\measuredangle B) = 45^\circ$ $\Bigg/ \Rightarrow \Delta ABC$ este Δ dr. isoscel $\Rightarrow$

$\Rightarrow AC = AB = 5$ cm.

Subiectul al II-lea.

1. $M=\begin{pmatrix}-2 & 2\\ -1 & -1\end{pmatrix}$, $I_2=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

a) $\det M = \begin{vmatrix}-2 & 2\\ -1 & -1\end{vmatrix} = 2+2=4 \Rightarrow \det M = 4$

b) $M\cdot M + 3M + 4I_2 = O_2$, $O_2=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$

$M\cdot M + 3M + 4I_2 = \begin{pmatrix}-2 & 2\\ -1 & -1\end{pmatrix}\begin{pmatrix}-2 & 2\\ -1 & -1\end{pmatrix} + 3\begin{pmatrix}-2 & 2\\ -1 & -1\end{pmatrix} + 4\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} =$

$=\begin{pmatrix}4-2 & -4-2\\ 2+1 & -2+1\end{pmatrix} + \begin{pmatrix}-6 & 6\\ -3 & -3\end{pmatrix} + \begin{pmatrix}4 & 0\\ 0 & 4\end{pmatrix} =$

$=\begin{pmatrix}2 & -6\\ 3 & -1\end{pmatrix} + \begin{pmatrix}-6 & 6\\ -3 & -3\end{pmatrix} + \begin{pmatrix}4 & 0\\ 0 & 4\end{pmatrix} = \begin{pmatrix}2-6+4 & -6+6+0\\ 3-3+0 & -1-3+4\end{pmatrix} =$

$=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} \Rightarrow M\cdot M + 3M + 4I_2 = O_2.$

c) $a, b = ?$ a.î. $M\cdot M\cdot M = aM + bI_2$.

$M\cdot M\cdot M = \begin{pmatrix}2 & -6\\ 3 & -1\end{pmatrix}\cdot\begin{pmatrix}-2 & 2\\ -1 & -1\end{pmatrix} = \begin{pmatrix}-4+6 & 4+6\\ -6+1 & 6+1\end{pmatrix} = \begin{pmatrix}2 & 10\\ -5 & 7\end{pmatrix}$

$M\cdot M = \begin{pmatrix}2 & -6\\ 3 & -1\end{pmatrix}$

$aM + bI_2 = a\begin{pmatrix}-2 & 2\\ -1 & -1\end{pmatrix} + b\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}-2a & 2a\\ -a & -a\end{pmatrix} + \begin{pmatrix}b & 0\\ 0 & b\end{pmatrix} =$

$=\begin{pmatrix}-2a+b & 2a\\ -a & -a+b\end{pmatrix}$

$M\cdot M\cdot M = aM + bI_2 \Leftrightarrow \begin{pmatrix}2 & 10\\ -5 & 7\end{pmatrix} = \begin{pmatrix}-2a+b & 2a\\ -a & -a+b\end{pmatrix} \Rightarrow$

$\Rightarrow \begin{cases}2a=10\\ -a+b=7\end{cases} \Rightarrow \begin{cases}a=5\\ b=7+5\end{cases} \Rightarrow \begin{cases}a=5\\ b=12\end{cases}$

2. $f = X^3 - 5X^2 + 5X - 1$

a) $f(1) = 0$

$f(1) = 1^3 - 5\cdot 1^2 + 5\cdot 1 - 1 = 1 - 5 + 5 - 1 = 0 \Rightarrow f(1) = 0.$

b) $f(a) + f(-a) + 2 \leq 0, \ \forall a \in \mathbb{R}.$

$f(a) = a^3 - 5a^2 + 5a - 1$

$f(-a) = (-a)^3 - 5\cdot(-a)^2 + 5\cdot(-a) - 1 = -a^3 - 5a^2 - 5a - 1.$

$f(a) + f(-a) + 2 = \cancel{a^3} - 5a^2 + \cancel{5a} - \cancel{1} - \cancel{a^3} - 5a^2 - \cancel{5a} - \cancel{1} + \cancel{2} = -10a^2$

$\Rightarrow f(a) + f(-a) + 2 = -10a^2 \leq 0, \ \forall a \in \mathbb{R}.$

c) $x_1^2 + x_2^2 + x_3^2 = 15\, x_1 x_2 x_3$

Relațiile lui Vieté:

$$\begin{cases} x_1 + x_2 + x_3 = -\dfrac{b}{a} = -\dfrac{(-5)}{1} = \dfrac{5}{1} = 5 \\ x_1x_2 + x_1x_3 + x_2x_3 = \dfrac{c}{a} = \dfrac{5}{1} = 5 \\ x_1x_2x_3 = -\dfrac{d}{a} = -\dfrac{(-1)}{1} = \dfrac{1}{1} = 1. \end{cases}$$

Știm că: $(x_1 + x_2 + x_3)^2 = x_1^2 + x_2^2 + x_3^2 + 2(x_1x_2 + x_1x_3 + x_2x_3) \Rightarrow$

$x_1^2 + x_2^2 + x_3^2 = (x_1 + x_2 + x_3)^2 - 2(x_1x_2 + x_1x_3 + x_2x_3)$

$x_1^2 + x_2^2 + x_3^2 = 5^2 - 2\cdot 5 = 25 - 10 = 15$

Pe 15 îl mai putem scrie ca $15\cdot 1$ adică $15\cdot x_1x_2x_3$.

Deci $x_1^2 + x_2^2 + x_3^2 = 15\cdot 1 = 15\cdot x_1x_2x_3$

Subiectul al III-lea.

1. $f:\mathbb{R}\to\mathbb{R}$, $f(x)=2x^3-6x+1$

a) $f'(x)=6(x-1)(x+1)$, $x\in\mathbb{R}$

$f'(x)=(2x^3-6x+1)'=(2x^3)'-(6x)'+(1)'=6x^2-6+0=$

$=6x^2-6=6(x^2-1)=6(x-1)(x+1) \Rightarrow$

$\Rightarrow f'(x)=6(x-1)(x+1)$

b) $x=1$

ec. tangentei: $y-f(x_0)=f'(x_0)(x-x_0)$, unde $x_0=1$.

$y-f(1)=f'(1)(x-1) \Rightarrow y+3=0\cdot(x-1)\Rightarrow y+3=0$

$f(1)=2\cdot1-6\cdot1+1=2-6+1=-3$

$f'(1)=6(1-1)(1+1)=6\cdot0\cdot2=0.$

$y+3=0 \Rightarrow y=-3$

c) $f(2012)+f(2014)\le f(2013)+f(2015)$

x	$-\infty$			-1			1			$+\infty$
$f(x)$		$\nearrow$		5	$\searrow$		-3	$\nearrow$		
$f'(x)$		$+$	$+$	0	$-$	$-$	0	$+$	$+$	

$f(1)=-3$

$f(-1)=2(-1)^3-6(-1)+1=-2+6+1=5$

$f'(x)\ge0,\ \forall x\in(-\infty;-1]\cup[1,+\infty) \Rightarrow$

$\Rightarrow f(x)$ crescătoare pe $(-\infty;-1]\cup[1,+\infty)$.

$\{2012, 2013, 2014, 2015\}\in[1,+\infty) \Rightarrow$

$\Rightarrow f(2012)\le f(2013)$ și $f(2014)\le f(2015) \Rightarrow$

$\Rightarrow f(2012)+f(2014)\le f(2013)+f(2015)$

2. $f:\mathbb{R}\to\mathbb{R}$, $f(x)=x^2-4$

a) $\int_0^1 (f(x)+4)\,dx = \int_0^1 (x^2-4+4)\,dx = \int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 =$

$= \frac{1}{3} - 0 = \frac{1}{3} \Rightarrow \int_0^1 (f(x)+4)\,dx = \frac{1}{3}$

b) $g:\mathbb{R}\to\mathbb{R}$, $g(x)=\frac{1}{f(x)+5}$, $x=0$ și $x=1$.

$A = \int_0^1 |g(x)|\,dx = \int_0^1 \frac{1}{x^2-4+5}\,dx = \int_0^1 \frac{1}{x^2+1}\,dx = \operatorname{arctg} x\Big|_0^1 =$

$= \operatorname{arctg} 1 - \operatorname{arctg} 0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}$.

c) $a=?$, $a>1$, $a\in\mathbb{R}$

$\int_1^a \frac{f(x)+4}{x}\,dx = 12$.

$\int_1^a \frac{f(x)+4}{x}\,dx = \int_1^a \frac{x^2+4-4}{x}\,dx = \int_1^a \frac{x^{\cancel{2}}}{\cancel{x}}\,dx = \int_1^a x\,dx =$

$= \frac{x^2}{2}\Big|_1^a = \frac{a^2}{2} - \frac{1}{2} = \frac{a^2-1}{2}$

$\int_1^a \frac{f(x)+4}{x}\,dx = 12 \Leftrightarrow \frac{a^2-1}{2} = 12 \Leftrightarrow a^2-1 = 24 \Leftrightarrow$

$\Leftrightarrow a^2-1-24=0 \Leftrightarrow a^2-25=0 \Leftrightarrow (a-5)(a+5)=0 \Rightarrow$

$\Rightarrow a-5=0 \Rightarrow a=5$ sau $a+5=0 \Rightarrow a=-5$ / $\Rightarrow$
$a>1$

$\Rightarrow$ soluția finală $a=5$

Pentru $a=-5$ nu convine, deoarece $a>1$ (din enunț)