Variamta 3 (2015 - M_ tehnologic)

Subiectul I

1. $a=16$, $b=9$

$m_g = \sqrt{a \cdot b} = \sqrt{16 \cdot 9} = \sqrt{144} = \sqrt{12^2} = 12 \Rightarrow m_g = 12$

2. $m = ?$ a.î. $f(2)=0$, $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x + m$

$f(2) = 2 + m$

$f(2) = 0 \Leftrightarrow 2 + m = 0 \Rightarrow m = -2$

3. $3^{2x+1} = 3^5 \Rightarrow 2x+1 = 5 \Leftrightarrow 2x = 5-1 \Leftrightarrow 2x = 4 \Rightarrow x = 2$

4. $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$M_2 = \{1 \cdot 2; 2 \cdot 2; 3 \cdot 2; 4 \cdot 2\} = \{2; 4; 6; 8\} \Rightarrow$

$\Rightarrow$ nr. cazurilor favorabile $= 4$

nr. cazurilor posibile $= 9$, deoarece mulțimea A este formată din 9 cifre (adică 9 elemente).

$P = \dfrac{\text{nr. cazurilor favorabile}}{\text{nr. cazurilor posibile}} = \dfrac{4}{9}$

5. $A(-1; 3)$, $B(5; 3)$

Fie $M(x_M, y_M)$ mijlocul segmentului $AB \Rightarrow$

$\Rightarrow M(x_M, y_M) = M\left(\dfrac{x_B + x_A}{2}; \dfrac{y_B + y_A}{2}\right) = M\left(\dfrac{5-1}{2}; \dfrac{3+3}{2}\right) =$

$= M\left(\dfrac{4}{2}; \dfrac{6}{2}\right) = M(2; 3) \Rightarrow x_M = 2$ și $y_M = 3$

6. $\sin^2 x + \cos^2 x = 1 \Leftrightarrow \sin^2 x = 1 - \cos^2 x \Leftrightarrow$

$\Leftrightarrow \sin^2 x = 1 - \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{4}{1} - \frac{3}{4} \Leftrightarrow \sin^2 x = \frac{1}{4}$

$x \in \left(0; \frac{\pi}{2}\right)$ $\Rightarrow$

$\Rightarrow \sin x = \sqrt{\frac{1}{4}} = \frac{1}{2} \Rightarrow \sin x = \frac{1}{2}$

Subiectul al II-lea

1. $A = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}$, $B = \begin{pmatrix} -4 & 0 \\ 0 & 4 \end{pmatrix}$; $C(x) = \begin{pmatrix} x & 1 \\ 2 & 3 \end{pmatrix}$, $x \in \mathbb{R}$

a) $\det A = \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = 1 \cdot 1 - 2 \cdot 3 = 1 - 6 = -5 \Rightarrow \det A = -5$

b) $\det(A + C(-1)) = \det B$

$A + C(-1) = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1-1 & 3+1 \\ 2+2 & 1+3 \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ 4 & 4 \end{pmatrix}$

$C(-1) = \begin{pmatrix} -1 & 1 \\ 2 & 3 \end{pmatrix}$

$\det(A + C(-1)) = \begin{vmatrix} 0 & 4 \\ 4 & 4 \end{vmatrix} = 0 \cdot 4 - 4 \cdot 4 = 0 - 16 = -16$

$\det B = \begin{vmatrix} -4 & 0 \\ 0 & 4 \end{vmatrix} = (-4) \cdot 4 - 0 \cdot 0 = -16$ $\Rightarrow$

$\Rightarrow \det(A + C(-1)) = \det B$

c) $x = ?$ a.î. $C(x) \cdot A - A \cdot C(x) = B$

$C(x) \cdot A = \begin{pmatrix} x & 1 \\ 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} x+2 & 3x+1 \\ 2+6 & 6+3 \end{pmatrix} = \begin{pmatrix} x+2 & 3x+1 \\ 8 & 9 \end{pmatrix}$

$A \cdot C(x) = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} x & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} x+6 & 1+9 \\ 2x+2 & 2+3 \end{pmatrix} = \begin{pmatrix} x+6 & 10 \\ 2x+2 & 5 \end{pmatrix}$

$C(X)\cdot A - A\cdot C(X) = B \Leftrightarrow \begin{pmatrix} x+2 & 3x+1 \\ 8 & 9 \end{pmatrix} - \begin{pmatrix} x+6 & 10 \\ 2x+2 & 5 \end{pmatrix} =$

$= \begin{pmatrix} -4 & 0 \\ 0 & 4 \end{pmatrix} \Leftrightarrow \begin{pmatrix} \cancel{x}+2-\cancel{x}-6 & 3x+1-10 \\ 8-2x-2 & 9-5 \end{pmatrix} = \begin{pmatrix} -4 & 0 \\ 0 & 4 \end{pmatrix} \Leftrightarrow$

$\Leftrightarrow \begin{pmatrix} -4 & 3x-9 \\ -2x+6 & 4 \end{pmatrix} = \begin{pmatrix} -4 & 0 \\ 0 & 4 \end{pmatrix} \Rightarrow 3x-9=0 \Leftrightarrow$

$\Leftrightarrow 3x = 9 \Rightarrow x = 9:3 \Rightarrow x = 3$

2. $f = x^3 + 2x^2 - 6x + 3$

a) $f(1) = 1^3 + 2\cdot 1^2 - 6\cdot 1 + 3 = 1+2-6+3 = 0 \Rightarrow f(1) = 0$

b)
$$\begin{array}{l|l} x^3+2x^2-6x+3 & x^2+3x-3 \\ \cline{2-2} -x^3-3x^2+3x & x-1 \\ \cline{1-1} \ /\ \ -x^2-3x+3 & \\ \quad\ \ x^2+3x-3 & \\ \cline{1-1} \quad\ \ /\ \ \ /\ \ \ / & \end{array}$$

$f : x^2+3x-3 = x-1$ rest 0

Deci, câtul este $x-1$, iar restul este 0

c) $x_1 + x_2 + x_3 + \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} = 0$

$\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} = \frac{x_2x_3 + x_1x_3 + x_1x_2}{x_1x_2x_3} = \frac{-6}{-3} = \frac{6}{3} = 2$

Relațiile lui Viète: $\begin{cases} x_1+x_2+x_3 = -\frac{b}{a} = -\frac{2}{1} = -2 \\ x_1x_2 + x_1x_3 + x_2x_3 = \frac{c}{a} = \frac{-6}{1} = -6 \\ x_1x_2x_3 = -\frac{d}{a} = -\frac{3}{1} = -3 \end{cases}$

$x_1+x_2+x_3 + \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} = -2+2 = 0 \Rightarrow$

$\Rightarrow x_1+x_2+x_3 + \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} = 0$

Subiectul al III-lea

1. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3 - 3x + 1$

a) $f'(x) = (x^3 - 3x + 1)' = (x^3)' - (3x)' + (1)' = 3x^2 - 3 + 0 =$
$= 3x^2 - 3 = 3(x^2 - 1) = 3(x-1)(x+1) \Rightarrow$

$\Rightarrow f'(x) = 3(x-1)(x+1)$, $x \in \mathbb{R}$

b) $\lim\limits_{x \to +\infty} \frac{f(x) - x^3}{x} = \lim\limits_{x \to +\infty} \frac{\cancel{x^3} - 3x + 1 - \cancel{x^3}}{x} =$

$= \lim\limits_{x \to +\infty} \frac{-3x + 1}{x} = \lim\limits_{x \to +\infty} \frac{\cancel{x}(-3 + \overbrace{\frac{1}{x}}^{\to 0})}{\cancel{x}} = -3$

c) $-1 \le f(x) \le 3$, $\forall x \in [-1, 1]$.

x	$-\infty$			-1				1			$+\infty$
$f(x)$				3				-1			
$f'(x)$	$+$	$+$	$+$	0	$-$	$-$	$-$	0	$+$	$+$	$+$

$f'(x) = 3(x-1)(x+1)$
$x - 1 = 0 \Rightarrow x = 1$
$x + 1 = 0 \Rightarrow x = -1$
$f'(1) = 0$
$f'(-1) = 0$
$f(-1) = (-1)^3 - 3 \cdot (-1) + 1 = -1 + 3 + 1 = 3$
$f(1) = 1^3 - 3 \cdot 1 + 1 = 1 - 3 + 1 = -1$
$f'(x) < 0$, $\forall x \in [-1, 1] \Rightarrow f$ este descrescătoare pe intervalul $[-1, 1] \Rightarrow -1 \le f(x) \le 3$, $\forall x \in [-1, 1]$

2. $f:(0,+\infty)\to\mathbb{R}$, $f(x)=2x+\frac{1}{x}$

a) $\int_2^3\left(f(x)-\frac{1}{x}\right)dx=\int_2^3\left(2x+\cancel{\frac{1}{x}}-\cancel{\frac{1}{x}}\right)dx=\int_2^3 2x\,dx=$

$=\cancel{2}\cdot\frac{x^2}{\cancel{2}}\Big|_2^3=x^2\Big|_2^3=3^2-2^2=9-4=5\Rightarrow$

$\Rightarrow\int_2^3\left(f(x)-\frac{1}{x}\right)dx=5$

b) $F:(0,+\infty)\to\mathbb{R}$, $F(x)=x^2+\ln x+2015$

$F(x)$ este o primitivă a lui $f\Leftrightarrow F'(x)=f(x)$

$F'(x)=(x^2+\ln x+2015)'=(x^2)'+(\ln x)'+(2015)'=$

$=2x+\frac{1}{x}+0=2x+\frac{1}{x}=f(x)\Rightarrow$

$\Rightarrow F'(x)=f(x),\ \forall x\in(0,+\infty)\Rightarrow$

$\Rightarrow F$ este o primitivă a funcției f.

c) $g:[1,2]\to\mathbb{R}$, $g(x)=f(x)-2x$

$V=\pi\int_1^2[g(x)]^2dx=\pi\int_1^2(f(x)-2x)^2dx=\pi\int_1^2\left(\cancel{2x}+\frac{1}{x}-\cancel{2x}\right)^2dx=$

$=\pi\int_1^2\left(\frac{1}{x}\right)^2dx=\pi\int_1^2\frac{1}{x^2}dx=\pi\cdot\int_1^2(x^{-2})dx=\pi\cdot\frac{x^{-2+1}}{-2+1}\Big|_1^2=$

$=\pi\cdot\frac{x^{-1}}{-1}\Big|_1^2=\pi\cdot(-x^{-1})\Big|_1^2=\pi\cdot\left(-\frac{1}{x}\right)\Big|_1^2=\left(-\frac{1}{2}+\frac{1}{1}\right)\pi=$

$=\frac{-1+2}{2}\cdot\pi=\frac{1}{2}\cdot\pi=\frac{\pi}{2}\Rightarrow V=\frac{\pi}{2}$