M_st-mat 2017

Simulare

Clasa a XII-a

Subiectul I

1. $z = ?$

$2z + \bar{z} = 6 + i$

Fie $z = a + bi$

$\bar{z} = a - bi$

$2z + \bar{z} = 2(a+bi) + (a-bi) = 2a + 2bi + a - bi =$

$= 3a + bi$

$3a + bi = 6 + i \Rightarrow \begin{cases} 3a = 6 \\ b = 1 \end{cases} \Leftrightarrow \begin{cases} a = 2 \\ b = 1 \end{cases}$

$z = a + bi = 2 + i$

2. $f: \mathbb{R} \to \mathbb{R},\ f(x) = 4x - 5$

$f(1) + f(2) + f(3) + \dots + f(10) = (4\cdot 1 - 5) + (4\cdot 2 - 5) +$

$+ (4\cdot 3 - 5) + \dots + (4\cdot 10 - 5) = 4(1+2+3+\dots+10) - 5\cdot 10 =$

$= \cancel{4}^{2} \cdot \frac{10(10+1)}{\cancel{2}_{1}} - 50 = 20 \cdot 11 - 50 = 220 - 50 = 170$

Aplicăm $1+2+3+4+\dots+10+\dots+n = \frac{n(n+1)}{2}$

3. $\log_2(x+3) = 1 + \log_2(x+1) \Leftrightarrow \log_2(x+3) = \log_2 2 + \log_2(x+1)$

$\Leftrightarrow \log_2(x+3) = \log_2[2(x+1)] \Rightarrow x + 3 = 2(x+1) \Rightarrow$

$\Rightarrow x + 3 = 2x + 2 \Rightarrow 2x - x = 3 - 2 \Rightarrow x = 1$

C.E. $\begin{cases} x+3 > 0 \\ x+1 > 0 \end{cases} \Rightarrow \begin{cases} x > -3 \\ x > -1 \end{cases} \Rightarrow S = \{1\}$

4. $P = ?$

nr. naturale de 2 cifre $= 99 - 10 + 1 = 89 + 1 = 90$

nr. naturale de 2 cifre identice $= \{11, 22, 33, 44, 55, 66, 77, 88, 99\} = 9$

$$P = \frac{9}{90} = \frac{1}{10}$$

5. $A = (1,1)$, $B(5,5)$, $C(-2,6)$

Ecuatia dreptei este: $y - y_0 = m(x - x_0)$

(x_0, y_0 sunt coordonatele punctului prin care trece dreapta)

(d) dreapta $\perp$ pe AB are panta: $m_d = -\frac{1}{m_{AB}}$.

$$m_d = \frac{-1}{\frac{y_B - y_A}{x_B - x_A}} = \frac{-1}{\frac{5-1}{5-1}} = \frac{-1}{\frac{4}{4}} = \frac{-1}{1} = -1.$$

$x_0 = -2$, $y_0 = 6$.

Ec dreptei: $y - 6 = -1 \cdot (x + 2) \Leftrightarrow$

$\Leftrightarrow y - 6 = -x - 2 \Leftrightarrow y = -x - 2 + 6 \Leftrightarrow$

$\Leftrightarrow y = -x + 4$

6. ΔABC, $AB = 3\sqrt{2}$, $m(\sphericalangle ACB) = 30^\circ$, $m(\sphericalangle BAC) = 45^\circ$.

$BC = ?$

Aplicăm teorema sinusului: $\frac{AB}{\sin C} = \frac{BC}{\sin A} \Rightarrow$

$$\Rightarrow BC = \frac{AB \cdot \sin A}{\sin C} = \frac{3\sqrt{2} \cdot \sin 45^\circ}{\sin 30^\circ} = \frac{3\sqrt{2} \cdot \frac{\sqrt{2}}{2}}{\frac{1}{2}} =$$

$$= \frac{6}{2} : \frac{1}{2} = \frac{6}{\cancel{2}} \cdot \frac{\cancel{2}}{1} = 6$$

Subiectul al II-lea

1. $A(x) = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & x \\ 4 & 9 & x^2 \end{pmatrix}$

a) $A(1) - A(0) = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 4 & 9 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 0 \\ 4 & 9 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

b) $\det(A(x)) = (x-2)(x-3), \forall x \in \mathbb{R}$

$$\det(A(x)) = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & x \\ 4 & 9 & x^2 \end{vmatrix} = 3x^2 + 18 + 4x - 12 - 2x^2 - 9x =$$
$$= x^2 - 5x + 6 =$$
$$= x^2 - 2x - 3x + 6 =$$
$$= x(x-2) - 3(x-2) =$$
$$= (x-2)(x-3), (\forall) x \in \mathbb{R}$$

c) $a = ?$

$\det(A(a)) \leq \det(A(x)), (\forall) x \in \mathbb{R}$

$\det(A(x)) = x^2 - 5x + 6$

Valoarea minimă a ec: $x^2 - 5x + 6$ este dată de formula $x = -\frac{b}{2a} = -\frac{-5}{2 \cdot 1} = \frac{5}{2}$

Cum $\det(A(a)) \leq \det(A(x)) \Rightarrow a^2 - 5a + 6 \leq x^2 - 5x + 6$

$\Rightarrow \left(a - \frac{5}{2}\right)^2 - \frac{1}{4} = \left(x - \frac{5}{2}\right)^2 - \frac{1}{4} \Rightarrow a = \frac{5}{2} = x$

(Ecuatia: $x^2 - 5x + 6$ are un minim egal cu $-\frac{\Delta}{4a} = -\frac{1}{4}$, care se realizează pentru $x = -\frac{b}{2a} = \frac{5}{2}$.)

2. $x \circ y = 4xy - 4x - 4y + 5$

a) $x \circ y = 4(x-1)(y-1)+1 = 4(xy - x - y + 1) + 1 =$
$= 4xy - 4x - 4y + 4 + 1 = 4xy - 4x - 4y + 5, (\forall) x, y \in \mathbb{R}$

b) $N = 2016 \circ 2017 = 4(2016-1)(2017-1)+1 =$
$= 4 \cdot 2015 \cdot 2016 + 1 = 4 \cdot 2015 \cdot (2015+1) + 1 =$
$= 4 \cdot 2015^2 + 4 \cdot 2015 + 1 = (2 \cdot 2015 + 1)^2 =$
$= (4030+1)^2 = 4031^2$

c) $a, b = ?$, $a, b \in \mathbb{N}$
$a \circ b = 13$
$a \circ b = 4(a-1)(b-1)+1$
$4(a-1)(b-1)+1 = 13 \Leftrightarrow 4(a-1)(b-1) = 13 - 1 \Leftrightarrow$
$\Leftrightarrow 4(a-1)(b-1) = 12 \quad / :4 \Leftrightarrow$
$\Leftrightarrow (a-1)(b-1) = 3 \Rightarrow (a-1)(b-1) = 3 \cdot 1$ sau
$(a-1)(b-1) = 1 \cdot 3$

caz 1. $(a-1)(b-1) = 3 \cdot 1 \Rightarrow \begin{cases} a-1=3 \\ b-1=1 \end{cases} \Rightarrow \begin{cases} a=4 \\ b=2 \end{cases}$

caz 2. $(a-1)(b-1) = 1 \cdot 3 \Rightarrow \begin{cases} a-1=1 \\ b-1=3 \end{cases} \Rightarrow \begin{cases} a=2 \\ b=4 \end{cases}$

Subiectul al III-lea

1. $f:(0,+\infty)$, $f(x) = x^2 \ln x$

a) $f'(x) = (x^2)' \cdot \ln x + x^2 (\ln x)' = 2x \ln x + x^{\cancel{2}} \cdot \frac{1}{\cancel{x}} = 2x \ln x + x =$

Aplicăm: $(f \cdot g)' = f' g + f \cdot g'$

$= x(2 \ln x + 1)$, $x \in (0, +\infty)$

b) Ec tangentei : $y - f(x_0) = f'(x_0)(x - x_0)$

$x_0 = 1.$

$f(1) = 1^2 \cdot \ln 1 = 1 \cdot 0 = 0$

$f'(1) = 1 \cdot (2 \ln 1 + 1) = 1 \cdot (2 \cdot 0 + 1) = 1$

$y - 0 = 1 \cdot (x - 1) \Rightarrow y = x - 1$

c) $1 + 2e f(x) \geq 0$,

Pentru $f'(x) = 0 \Leftrightarrow x(2\ln x + 1) = 0 \Rightarrow x = 0 \notin (0, +\infty)$

$\Rightarrow 2\ln x + 1 = 0$

$2\ln x + 1 = 0 \Leftrightarrow 2\ln x = -1 \Leftrightarrow \ln x^2 = -\ln e \Leftrightarrow$

$\Leftrightarrow \ln x^2 = \ln e^{-1} \Rightarrow x^2 = e^{-1} \Rightarrow x^2 = \frac{1}{e} \Rightarrow$

$\Rightarrow x = \sqrt{\frac{1}{e}} = \frac{1}{\sqrt{e}}$

Pentru $x \in \left(0; \frac{1}{\sqrt{e}}\right) \Rightarrow f'(x) \leq 0 \Rightarrow f$ descrescătoare pe $\left(0, \frac{1}{\sqrt{e}}\right]$.

Pentru $x \in \left[\frac{1}{\sqrt{e}}, +\infty\right) \Rightarrow f'(x) \geq 0 \Rightarrow f$ crescătoare pe $\left[\frac{1}{\sqrt{e}}, +\infty\right)$

$f\left(\frac{1}{\sqrt{e}}\right) = \left(\frac{1}{\sqrt{e}}\right)^2 \ln \frac{1}{\sqrt{e}} = \frac{1}{e} \ln e^{-\frac{1}{2}} = \frac{1}{e} \cdot \left(-\frac{1}{2}\right) = -\frac{1}{2e}$.

Cum $f\left(\frac{1}{\sqrt{e}}\right) = -\frac{1}{2e} \Rightarrow f(x) \geq -\frac{1}{2e} \Leftrightarrow 2e f(x) \geq -1 \Leftrightarrow$

$\Leftrightarrow 1 + 2e f(x) \geq 0, (\forall) x \in (0, +\infty)$

2. $f:\mathbb{R}\to\mathbb{R}$, $f(x)=(x-1)e^x$

a) $\int_0^1 f(x)e^{-x}dx = \int_0^1 (x-1)e^x\cdot e^{-x}dx = \int_0^1 (x-1)e^{x-x}dx =$

$= \int_0^1 (x-1)e^0 dx = \int_0^1 (x-1)dx = \int_0^1 x\,dx - \int_0^1 dx =$

$= \left(\frac{x^2}{2}-x\right)\Big|_0^1 = \frac{1}{2}-\frac{2}{1}-0 = -\frac{1}{2}$

b) $a=?$, $F:\mathbb{R}\to\mathbb{R}$, $F(x)=(x+a)e^x$

$F(x)$ primitivă a lui $f(x) \Rightarrow F'(x)=f(x)$

$F'(x)=[(x+a)e^x]' = (x+a)'\cdot e^x+(x+a)\cdot(e^x)' =$

$= e^x+(x+a)e^x = e^x(1+x+a)$

$e^x(1+x+a) = (x-1)e^x \;/:e^x \Rightarrow 1+\cancel{x}+a=\cancel{x}-1 \Rightarrow$

$\Rightarrow a=-1-1 \Rightarrow a=-2$

c) $\int_0^1 x^3 f(x)dx \le -\frac{1}{20}$

$\int_0^1 x^3 f(x)dx = \int_0^1 x^3\cdot(x-1)\cdot e^x dx = \int_0^1 (x^4-x^3)e^x dx$

$x^3 f(x) = (x^4-x^3)e^x$, $x\in[0,1]$ $\Rightarrow$ $1\le e^x$, $x^4-x^3\le 0$, $x^3f(x)=(x^4-x^3)e^x$ $\Rightarrow x^3f(x)\le x^4-x^3$

$\int_0^1 x^3 f(x)dx \le \int_0^1 (x^4-x^3)dx = \int_0^1 x^4dx - \int_0^1 x^3 dx = \left(\frac{x^5}{5}-\frac{x^4}{4}\right)\Big|_0^1 =$

$= \frac{1^4}{5}-\frac{1^5}{4} = -\frac{1}{20} \Rightarrow \int_0^1 x^3 f(x)dx \le -\frac{1}{20}$