Științe ale naturii

Varianta 2

2017

Subiectul I

1. $a_1 = 4,\ a_2 = 7,\ a_3 = ?$

$a_2 = a_1 + r$

$7 = 4 + r \Rightarrow r = 7 - 4 = 3.$

$a_n = a_1 + (n-1)r$

$a_3 = a_1 + (3-1)\cdot 3 = 4 + 2\cdot 3 = 4 + 6 = 10$

2. $x^2 - 4x + 1 = 0$

$4x_1x_2 - (x_1 + x_2) = 0.$

Rel. lui Viete

$$\Rightarrow \begin{cases} x_1 + x_2 = -\frac{b}{a} \\ x_1x_2 = \frac{c}{a} \end{cases} \Leftrightarrow \begin{cases} x_1 + x_2 = -\frac{-4}{1} = +4 \\ x_1x_2 = \frac{1}{1} = 1 \end{cases}$$

$4x_1x_2 - (x_1 + x_2) = 4\cdot 1 - 4 = 4 - 4 = 0.$

3. $2^{2x+1} = \frac{1}{8} \Leftrightarrow 2^{2x+1} = 8^{-1} \Leftrightarrow 2^{2x+1} = 2^{-3} \Rightarrow$

$\Rightarrow 2x + 1 = -3 \Leftrightarrow 2x = -3 - 1 \Leftrightarrow 2x = -4 \Rightarrow x = -\frac{4}{2} = -2$

4. $P = ?$

numere naturale de 2 cifre $= 99 - 10 + 1 = 89 + 1 = 90$ de numere

$M_{15} = \{1\cdot 15, 2\cdot 15, 3\cdot 15, 4\cdot 15, 5\cdot 15, 6\cdot 15\} = \{15, 30, 45, 60, 75, 90\}$

$P = \frac{\text{nr. cazurilor favorabile}}{\text{nr. cazurilor posibile}} = \frac{6^{\,16}}{90} = \frac{1}{15}$

5. $a = ?$, A, B, C - coliniare

$A(0,1)$, $B(1,1)$, $C(3,a)$

Trei puncte sunt coliniare $\Leftrightarrow$ $\det(A,B,C) = 0$, adică

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 3 & a & 1 \end{vmatrix} = 0 \Leftrightarrow 0 + a + \cancel{3} - \cancel{3} - 1 - 0 = 0 \Leftrightarrow$$

$$\Leftrightarrow a - 1 = 0 \Rightarrow a = 1$$

sau

$$m_{AB} = m_{AC} \Leftrightarrow 0 = \frac{a-1}{3} \Leftrightarrow a - 1 = 0 \Rightarrow a = 1.$$

m = panta

$$m_{AB} = \frac{y_B - y_A}{x_B - x_A} = \frac{1-1}{1-0} = \frac{0}{1} = 0$$

$$m_{AC} = \frac{y_C - y_A}{x_C - x_A} = \frac{a-1}{3-0} = \frac{a-1}{3}$$

6. $\triangle ABC$

$AB = 4\sqrt{3}$, $AC = 4$, $\sin C = \frac{\sqrt{3}}{2}$

$\sin B = ?$

Din teorema sinusului avem: $\frac{AB}{\sin C} = \frac{AC}{\sin B} \Rightarrow$

$$\Rightarrow \sin B = \frac{AC \cdot \sin C}{AB} = \frac{\cancel{4}^{2} \cdot \frac{\sqrt{3}}{\cancel{2}}}{4\sqrt{3}} = \frac{2\sqrt{3}}{\cancel{4}_{2}\sqrt{3}} = \frac{1}{2}$$

Subiectul al II-lea

1. $A(x) = \begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix}$

a) $\det(A(1)) = -1$.

$\det(A(x)) = \begin{vmatrix} 0 & x \\ x & 0 \end{vmatrix} = 0 - x^2 = -x^2$

$\det(A(1)) = -(1^2) = -1$.

b) $A(x) \cdot A(y) = xy I_2$, $(\forall)\ x, y \in \mathbb{R}$, $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$A(y) = \begin{pmatrix} 0 & y \\ y & 0 \end{pmatrix}$

$A(x) \cdot A(y) = \begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & y \\ y & 0 \end{pmatrix} = \begin{pmatrix} xy & 0 \\ 0 & xy \end{pmatrix} = xy \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} =$

$= xy \cdot I_2$.

c) $a = ?$, $A(3^a) \cdot A(3^{a+1}) \cdot A(3^{a+2}) = A(27)$

$A(3^a) \cdot A(3^{a+1}) = (3^a \cdot 3^{a+1}) \cdot I_2 = 3^{a+a+1} \cdot I_2 = 3^{2a+1} \cdot I_2 =$

$= 3^{2a+1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3^{2a+1} & 0 \\ 0 & 3^{2a+1} \end{pmatrix}$

$\begin{pmatrix} 3^{2a+1} & 0 \\ 0 & 3^{2a+1} \end{pmatrix} \cdot A(3^{a+2}) = \begin{pmatrix} 3^{2a+1} & 0 \\ 0 & 3^{2a+1} \end{pmatrix} \cdot \begin{pmatrix} 0 & 3^{a+2} \\ 3^{a+2} & 0 \end{pmatrix} =$

$= \begin{pmatrix} 0 & 3^{2a+1} \cdot 3^{a+2} \\ 3^{2a+1} \cdot 3^{a+2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 3^{2a+1+a+2} \\ 3^{2a+1+a+2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 3^{3a+3} \\ 3^{3a+3} & 0 \end{pmatrix} =$

$= \begin{pmatrix} 0 & 3^{3(a+1)} \\ 3^{3(a+1)} & 0 \end{pmatrix} = A(3^{3(a+1)})$

$A\left(3^{3(a+1)}\right) = A(27) \Leftrightarrow \begin{pmatrix} 0 & 3^{3(a+1)} \\ 3^{3(a+1)} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 27 \\ 27 & 0 \end{pmatrix} \Leftrightarrow$

$\Leftrightarrow \begin{pmatrix} 0 & 3^{3(a+1)} \\ 3^{3(a+1)} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 3^3 \\ 3^3 & 0 \end{pmatrix} \Rightarrow$

$\Rightarrow 3(a+1) = 3 \; /:3 \Rightarrow a+1 = 1 \Rightarrow a = 1-1 = 0$

2. $f = X^3 + mX^2 + 2X - 4, \; m \in R$

a) $m = 1, \; f(1) = 0 \,?$

$f(x) = X^3 + X^2 + 2X - 4$

$f(1) = 1^3 + 1^2 + 2 - 4 = 1+1+2-4 = 4-4 = 0.$

b) $X+2 \,/\, f \Rightarrow f(-2) = 0 \Leftrightarrow (-2)^3 + m(-2)^2 + 2\cdot(-2) - 4 = 0 \Leftrightarrow$

$\Leftrightarrow -8 + 4m - 4 - 4 = 0 \Leftrightarrow -16 + 4m = 0 \overset{/:4}{\Leftrightarrow} -4 + m = 0 \Rightarrow$

$\Rightarrow m = 4$

Pentru $m = 4 \Rightarrow f(X) = X^3 + 4X^2 + 2X - 4$

$f(-3) = (-3)^3 + 4\cdot(-3)^2 + 2\cdot(-3) - 4 = -27 + 36 - 6 - 4 =$

$= 9 - 10 = -1$

c) $m = ?$

$\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + x_1 + x_2 + x_3 = \frac{1}{2}$

Din relațiile lui Viete $\Rightarrow \begin{cases} x_1 + x_2 + x_3 = -\frac{b}{a} = -\frac{m}{1} = -m \\ x_1x_2 + x_2x_3 + x_1x_3 = \frac{c}{a} = 2 \\ x_1x_2x_3 = -\frac{d}{a} = -(-4) = 4 \end{cases}$

$\frac{1}{x_1}^{(x_2x_3} + \frac{1}{x_2}^{(x_1x_3} + \frac{1}{x_3}^{(x_1x_2} = \frac{x_2x_3 + x_1x_3 + x_1x_2}{x_1x_2x_3} = \frac{2}{4}^{(2} = \frac{1}{2}$

$NC = x_1x_2x_3$

$$\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+x_1+x_2+x_3=\frac{1}{2} \Leftrightarrow \frac{1}{2}-m=\frac{1}{2} \Leftrightarrow$$

$$\Leftrightarrow -m=\frac{1}{2}-\frac{1}{2} \Rightarrow -m=0 \Rightarrow m=0$$

Subiectul al III - lea

1. $f:\mathbb{R}\to\mathbb{R}$, $f(x)=\dfrac{x+2017}{e^x}$

a) $f'(x)=\dfrac{-(x+2016)}{e^x}$, $x\in\mathbb{R}$

notăm $g(x)=x+2017$

$h(x)=e^x$

$$\left(\frac{g}{h}\right)'=\frac{g'\cdot h-g\cdot h'}{h^2}$$

$$\left(\frac{x+2017}{e^x}\right)'=\frac{(x+2017)'\cdot e^x-(x+2017)\cdot(e^x)'}{(e^x)^2}=$$

$$=\frac{1\cdot e^x-(x+2017)\cdot e^x}{e^{2x}}=\frac{\cancel{e^x}(1-x-2017)}{e^{\cancel{2x}}}=\frac{-x-2016}{e^x}=$$

$$=\frac{-(x+2016)}{e^x}$$

b) Ecuația tangentei la graficul funcției f este:

$$\boxed{y-f(x_0)=f'(x_0)(x-x_0)}$$

$x_0=0$

$$f(0)=\frac{0+2017}{e^0}=\frac{2017}{1}=2017$$

$$f'(0)=\frac{-(0+2016)}{e^0}=\frac{-2016}{1}=-2016$$

$y - 2017 = -2016(x-0)$
$y = -2016x + 2017$

c) f convexă pe $[-2015, +\infty)$

f este convexă dacă $f'' \geq 0, (\forall)\, x \in [-2015, +\infty)$

$$f'(x) = \frac{-(x+2016)}{e^x}$$

$$f''(x) = \frac{[-(x+2016)]' \cdot e^x - [-(x+2016)] \cdot (e^x)'}{(e^x)^2} =$$

$$= \frac{-1 \cdot e^x + (x+2016) \cdot e^x}{(e^x)^2} = \frac{\cancel{e^x}(-1+x+2016)}{(e^x)^{\cancel{2}}} =$$

$$= \frac{x+2015}{e^x} = f''(x)$$

x	$-\infty$				-2015					$+\infty$
$x+2015$	$-$	$-$	$-$	$-$	0 $+$	$+$	$+$	$+$	$+$	
e^x	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	
$f''(x)$	$-$	$-$	$-$	$-$	0 $+$	$+$	$+$	$+$	$+$	$+$

din tabel $\Rightarrow f''(x) \geq 0$ când $x \in [-2015, +\infty) \Leftrightarrow$

$\Rightarrow f$ este convexă pe $[-2015, +\infty)$

2. $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{1}{x^2+1}$

a) $$\int_0^1 \frac{1}{f(x)}\,dx = \int_0^1 \frac{1}{\frac{1}{x^2+1}}\,dx = \int_0^1 (x^2+1)\,dx =$$

$$= \int_0^1 x^2\,dx + \int_0^1 dx = \frac{x^3}{3}\Big|_0^1 + x\Big|_0^1 = \frac{1^3}{3} - 0 + 1 - 0 =$$

$$= \frac{1}{3} + \frac{^{3)}1}{1} = \frac{1+3}{3} = \frac{4}{3}$$

b) $F(x) = ?$

$F(1) = \frac{\pi}{4} + 1$

$F(x) = \int f(x)\,dx = \int \frac{1}{x^2+1}\,dx = \text{arctg}\, x + C$

$\boxed{\text{Aplicăm formula } \int \frac{dx}{x^2+a^2} = \frac{1}{a}\,\text{arctg}\,\frac{x}{a} + C}$

$F(1) = \text{arctg}\, 1 + C$

$F(1) = \frac{\pi}{4} + 1$ $\Rightarrow C = 1.$

Deci $F(x) = \text{arctg}\, x + 1$

c) $n = ?$, $\int_0^n x f(x)\,dx = \frac{1}{2}\ln 5$

$\int x f(x)\,dx = \int x \cdot \frac{1}{x^2+1}\,dx = \frac{1}{2}\int \frac{2x}{x^2+1}\,dx =$

$= \frac{1}{2}\int \frac{(x^2+1)'}{x^2+1}\,dx = \frac{1}{2}\ln(x^2+1)$

$\boxed{\text{Aplicăm formula } \int \frac{u'}{u}\,du = \ln|u| + C}$

la noi $x^2+1 > 0.$

$\int_0^n x f(x)\,dx = \frac{1}{2}\ln(x^2+1)\Big|_0^n = \frac{1}{2}\ln(n^2+1) -$

$- \frac{1}{2}\ln(0+1) = \frac{1}{2}\ln(n^2+1) - \frac{1}{2}\underbrace{\ln 1}_{0} = \frac{1}{2}\ln(n^2+1)$

$\frac{1}{2}\ln(n^2+1) = \frac{1}{2}\ln 5 \Leftrightarrow \ln(n^2+1) = \ln 5 \Leftrightarrow$

$\Leftrightarrow n^2+1 = 5 \Leftrightarrow n^2+1-5 = 0 \Leftrightarrow n^2-4 = 0 \Leftrightarrow$

$\Leftrightarrow (n-2)(n+2) = 0 \Rightarrow n_1 = 2$, $n_2 = -2$ nu convine $\Rightarrow n = 2$